Math 249 Lecture 11 Notes

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1 The Frobenius Characteristic Map and the Casimir Element

At this point, we have 4 bases for the symmetric functions:

1. the monomial symmetric functions m_λ
2. the elementary symmetric functions e_λ
3. the homogeneous symmetric functions h_λ
4. the power-sum symmetric functions p_λ (if the coefficients contain $\mathbb{Q}$).

We showed that this 4th basis has some connection to the symmetric group.

1.1 The Frobenius characteristic map

Definition 1.1. The *Frobenius characteristic map* F : {characters of S_n} $\to \Lambda_{(n)}$ is the map

$$\chi \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma) p_{\gamma(\sigma)},$$

where $\gamma(\sigma)$ is the cycle partition of the permutation σ. Equivalently, we can say

$$\chi \mapsto \sum_{|\lambda|=n} \chi(\sigma_\lambda) p_\lambda \frac{|C_\lambda|}{n!} = \sum_{|\lambda|=n} \chi(\sigma_\lambda) \frac{p_\lambda}{z_\lambda},$$

where $|C_\lambda| = n!/z_\lambda$.

Example 1.1. Let $\mathbb{1}_{S_n}$ be the character of the trivial representation of S_n. Then

$$F(\mathbb{1}_{S_n}) = h_n.$$

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Recall the bijection $\omega : \Lambda \to \Lambda$ that sends $h_n \to e_n$ and $e_n \to h_n$. Applying ω to $H(t)$, the generating function for the h_n, we get

$$\omega H(t) = E(t) = \frac{1}{H(-t)}.$$

What ω does to the p_k is determined by its action on the generating function $H(t)$. Recall that

$$H(t) = \exp\left(\sum_k \frac{p_k t^k}{k}\right).$$

We have

$$\omega H(t) = \frac{1}{H(-t)} = \exp\left(-\sum_k \frac{p_k(-t)^k}{k}\right) = \exp\left(\sum_k \frac{(-1)^{k-1} p_k t^k}{k}\right),$$

which gives us that $\omega(p_k) = (-1)^{k-1} p_k$. Consequently, $\omega(p_\lambda) = \varepsilon(\sigma_\lambda) p_\lambda$. This means that

$$e_n = \sum_{|\lambda|=n} \varepsilon(\sigma_\lambda) \frac{p_\lambda}{z_\lambda},$$

which shows that

$$F(\varepsilon) = e_n.$$

In fact, for any character χ of S_n,

$$F(\chi \otimes \varepsilon) = \omega F(\chi).$$

1.1.1 Inner products of symmetric functions

Example 1.2. Let δ_λ be the character that is 1 on the conjugacy class of elements with cycle structure λ and 0 otherwise. Then

$$F(\delta_\lambda) = \frac{p_\lambda}{z_\lambda}.$$

How does F relate to inner products? The δ_λ form a basis for characters of S_n, and their inner product is

$$(\delta_\lambda, \delta_\mu) = \frac{1}{n!} \sum_{\sigma \in S_n} \delta_\lambda(\sigma) \delta_\mu(\sigma^{-1}) = \frac{|C_\lambda|}{n!} \delta_{\lambda,\mu} = \frac{1}{z_\lambda} \delta_{\lambda,\mu},$$

where $\delta_{\lambda,\mu}$ is a Kronecker delta. We then define an inner product on symmetric functions as follows.

Definition 1.2. The *inner product of symmetric functions* is defined on the power-sum basis as

$$\langle p_\lambda, p_\mu \rangle := z_\lambda \delta_{\lambda\mu}.$$

This definition makes F an isometry because

$$\langle F(\delta_\lambda), F(\delta_\mu)) = \langle p_\lambda / z_\lambda, p_\mu / z_\mu \rangle = \frac{1}{z_\lambda} \delta_{\lambda\mu}.$$

Also, for any character χ,

$$\chi(\sigma_\lambda) = z_\lambda (\chi, \delta_\lambda) = z_\lambda \langle F(\chi), F(\delta_\lambda) \rangle$$

This says that the $\{p_\lambda\}$ and $\{p_\lambda / z_\lambda\}$ are dual bases.

1.2 The Casimir element of a vector space

Definition 1.3. Let V be a finite dimensional inner product space. Two bases $\{u_i\}, \{v_j\}$ are *dual bases* if $\langle u_i, v_j \rangle = \delta_{i,j}$.

Definition 1.4. Let V be a finite dimensional inner product space with dual bases $\{u_i\}, \{v_j\}$. The *Casimir element* $\theta \in V \otimes V$ is $\theta = \sum_i u_i \otimes v_i$.

Remarkably, this depends only on $\langle \cdot, \cdot \rangle$. Since V is finite dimensional, we have the isomorphism $V \cong V^*$ given by $u_i \mapsto \xi_i$, where $\xi_j(u_i) = \delta_{i,j}$. Then we have

$$V \otimes V \cong V \otimes V^* \cong \mathrm{End(V)},$$

where the second isomorphism is given by $v \otimes \xi \mapsto (w \mapsto \xi(w)v)$.

What corresponds to the identity element $1_V \in \mathrm{End}(V)$? It is $\sum_i u_i \otimes \xi_i$, where $\xi_i(u_j) = \delta_{i,j}$. Then the Casimir element is the corresponding element in $V \otimes V$.

1.2.1 The Casimir element of the symmetric functions

For convenience of notation, call

$$\Omega = H(1) = \sum_{n=0}^{\infty} h_n = \prod_i \frac{1}{1 - x_i}.$$

We also call this $\Omega[X]$ to emphasize that this is a symmetric function in variables $x_1, x_2, \ldots$.

$$\Omega = \exp\left(\sum_{k>0} \frac{p_k}{k} \right).$$

$\Lambda_{\mathbb{Q}} \otimes \Lambda_{\mathbb{Q}}$ is isomorphic to the symmetric functions in $x_1, x_2, \ldots$ and $y_1, y_2, \ldots$, where the functions are symmetric separately with respect to the x_i and the y_j. So we may identify the Casimir element as

$$\theta_n = \sum_{|\lambda|=n} \frac{p_\lambda(X)p_\lambda(Y)}{z_\lambda}.$$

What is this? Note that

$$p_k(X)p_k(Y) = \left(\sum_i x_i^k\right)\left(\sum_j y_j^k\right) = \sum_{i,j}(x_i y_j)^k = p_k(XY),$$

which makes

$$p_\lambda(X)p_\lambda(Y) = p_{\lambda_1}(X)p_{\lambda_1}(Y)\cdots p_{\lambda_\ell}(X)p_{\lambda_\ell}(Y) = p_{\lambda_1}(X)\cdots p_{\lambda_\ell}(XY) = p_\lambda(XY).$$

We then get the following expression for the Casimir element:

$$\theta = \sum_n \sum_{|\lambda|=n} \frac{p_\lambda(XY)}{z_\lambda} = \Omega[XY] = \prod_{i,j} \frac{1}{1 - x_i y_j}.$$